

Proof of the Super Efimov Effect

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Abstract

We consider the system of 3 nonrelativistic spinless fermions in two dimensions, which interact through spherically-symmetric pair interactions. Recently a claim has been made for the existence of the so-called super Efimov effect [Y. Nishida *et al.*, Phys. Rev. Lett. 110, 235301 (2013)]. Namely, if the interactions in the system are fine-tuned to a p-wave resonance, an infinite number of bound states appears, whose negative energies are scaled according to the double exponential law. We present the mathematical proof that such system indeed has an infinite number of bound levels. We also prove that $\lim_{E \rightarrow 0} |\ln |\ln E||^{-1} N(E) = 8/(3\pi)$, where $N(E)$ is the number of bound states with the energy less than $-E < 0$. The value of this limit is equal exactly to the value derived in [Y. Nishida *et al.*] using renormalization group approach. Our proof resolves a recent controversy about the validity of results in [Y. Nishida *et al.*].

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I. INTRODUCTION

Efimov effect first discovered by V. Efimov in [1] is one of the most intriguing phenomena in physics. This effect appears in 3-body systems in 3-dimensional space, which interact through short-range pair-potentials. It is always possible to tune the couplings of the interactions in such a way that none of the particle pairs has a negative energy bound state, but at least two pairs have a resonance at zero energy. In this case the 3-body system exhibits an infinite sequence of bound levels, where the energy of the n -th level decreases exponentially with n . The rigorous proof of this effect in [2, 3] is a highlight of mathematical physics. Suppose that three particles are identical, the pair interaction is tuned to the zero energy resonance, and let N_E be the total number of 3-body bound states with the energy less than $-E < 0$. Then $\lim_{E \rightarrow 0} |\ln E|^{-1} N_E = s_0/(2\pi)$, where s_0 is the root of the known transcendental equation expressed in elementary functions [3].

Relatively recently the authors in [4] considered the system of 3 spinless fermions in flatland using field-theoretical methods. The spherically symmetric pair interaction of fermions was tuned in such a way that pairs of fermions had no negative spectrum but were at the coupling constant threshold [6, 7], so that a negligible increase of the coupling constant would result in the formation of an antisymmetric 2-body bound state with negative energy. In this case one says that the interactions are tuned to the zero energy p-wave resonance. In [4] the authors came to the conclusion that such system has two infinite series of bound states each corresponding to the orbital angular momentum $L = \pm 1$. The energies of these bound states E_n for large n were predicted to approach the form $E_n \sim -\exp(-2e^{\frac{3\pi n}{4} + \theta})$, where θ is a constant defined modulo $3\pi/4$. The authors termed this phenomenon "super Efimov effect". If N_E is the total number of 3-body bound states with the energy less than $-E < 0$ (for all values of the angular momentum) then the results in [4] predict that

$$\lim_{E \rightarrow 0} |\ln |\ln E||^{-1} N_E = 8/(3\pi). \quad (1)$$

There are two interesting features about the super Efimov effect. First, it turns out that the system of 3 spinless fermions in two dimensions may have an infinite number of bound states, though the same system in 3 dimensions has at most a finite number of levels with negative energy [8]. Secondly, the energy of the n -th level goes extremely fast to zero with increasing n . This is reflected in the double logarithm in (1) and differs from the Efimov

effect of 3 bosons in 3-dimensional space, where a single logarithm enters the similar formula [3].

Recently in the physics literature there were raised doubts about whether the super Efimov effect is real [9–11]. In [10] it was claimed that the sequence of levels with double exponential scaling does not exist and instead there emerges another infinite sequence of levels, which approaches the scaling law $E_n \sim -\exp(n^2\pi^2/2Y)$ with $Y > 0$ being a non-universal constant. The findings in [10] are in contradiction with Eq. (1). In [11] the authors observed the super Efimov effect in the lowest order of the hyperspherical expansion. However, the value of the limit in (1) was found to be $2(16/9 - 1/4)^{-1/2}$; the inclusion of higher order effects could not provide definitive conclusions on whether the infinite sequence of levels exists. In the present paper we shall provide a rigorous mathematical proof of (1). Hence, we demonstrate that the super Efimov effect is indeed real and the constant on the rhs of (1) coincides exactly with the one predicted in [4].

The basic idea behind the proof of (1) stems from [2], namely, one uses symmetrized Faddeev equations and the Birman-Schwinger principle [5–7, 12] for counting eigenvalues. Like in [2] we reduce the problem to counting the eigenvalues in the interval $(0, \infty)$ of an integral operator, which depends on the energy. Let us explain, however, the major difference. In [2] when the energy approached zero this integral operator approached (in the strong sense) a bounded integral operator, which had a nonempty essential spectrum in the interval $(1, \infty)$. In the 2-dimensional case a similar integral operator maintains discrete spectrum but its norm goes to infinity when the energy goes to zero. The control of appearing error terms becomes challenging because their norm diverges as well.

We shall use the following notations. An abstract Hilbert space \mathcal{H} is assumed to be separable, $\mathcal{C}(\mathcal{H})$ denotes the ideal of all compact operators on \mathcal{H} . For a self-adjoint operator $A \in \mathcal{C}(\mathcal{H})$ we denote by $\lambda_1(A), \lambda_2(A), \dots$ its non-negative eigenvalues (counting multiplicities) in descending order; if this sequence terminates at n_0 we set $\lambda_{n_0+1}(A) = \lambda_{n_0+2}(A) = \dots = 0$. For a self-adjoint operator A on \mathcal{H} we shall denote by $D(A)$, $\sigma(A)$ and $\sigma_{ess}(A)$ the domain, the spectrum, and the essential spectrum of A respectively [15]. $A \geq 0$ means that $(f, Af) \geq 0$ for all $f \in D(A)$, while $A \not\geq 0$ means that there exists $f_0 \in D(A)$ such that $(f_0, Af_0) < 0$. $\mathbf{n}(A, a)$ is the number of eigenvalues of A (counting multiplicities) that are larger than $a > 0$. By $\mu_n(A)$ we denote singular values of $A \in \mathcal{C}(\mathcal{H})$ listed in descending order [13]. Similarly, $\mathbf{n}_\mu(A, a) = \mathbf{n}(|A|, a)$ is the number of singular values of $A \in \mathcal{C}(\mathcal{H})$

that are larger than $a > 0$. $\|A\|_{HS}$ is the Hilbert-Schmidt norm of an operator A . For an interval $\Omega \subset \mathbb{R}$ the function $\chi_\Omega : \mathbb{R} \rightarrow \mathbb{R}$ is such that $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ otherwise. $\text{diag}\{a_1, a_2, a_3\}$ denotes a 3×3 matrix with the diagonal entries a_1, a_2, a_3 and zero off-diagonal elements.

II. MAIN RESULT

We shall consider 3 spinless fermions in \mathbb{R}^2 that interact through $v(|r_i - r_k|) \leq 0$, where r_i are particle position vectors. For pair interactions we assume that v is a Borel function, $|v(x)| \leq \alpha_1 e^{-\alpha_2|x|}$ with $\alpha_{1,2} > 0$ being constants. Regarding the fermion's mass m we shall use the units, where $\hbar^2/m = 1$. The Hamiltonian of this system reads

$$H = H_0 + \sum_{1 \leq i < k \leq 3} v(|r_i - r_k|), \quad (2)$$

where H_0 is the kinetic energy operator with the removed center of mass motion. Due to the Pauli principle H should be considered on an antisymmetrized space, which is constructed below. For $k = 1, 2, 3$ let $x_k, y_k \in \mathbb{R}^2$ be three sets of Jacobi coordinates, which are shown in Fig. 1

$$x_k = r_i - r_j \quad (3)$$

$$y_k = \frac{2}{\sqrt{3}} \left[r_k - \frac{r_i + r_j}{2} \right], \quad (4)$$

where (k, i, j) is an odd permutation of $(1, 2, 3)$. The scalings are chosen so that in all coordinate sets $H_0 = -\Delta_{x_k} - \Delta_{y_k}$. The coordinate sets are connected through the orthogonal linear transformation

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \quad (5)$$

where (i, k, j) is an odd permutation of $(1, 2, 3)$. Let us write Jacobi coordinates as functions of particle position vectors, that is $x_k = x_1(r_1, r_2, r_3)$ and $y_k = y_1(r_1, r_2, r_3)$. And let $p = (p(1), p(2), p(3))$ be a permutation of $(1, 2, 3)$. Then by definition $p(x_1) = x_1(r_{p(1)}, r_{p(2)}, r_{p(3)})$ and $p(y_1) = y_1(r_{p(1)}, r_{p(2)}, r_{p(3)})$. We define the action of the permutation operator p on $L^2(\mathbb{R}^4)$ as $pf(x_1, y_1) = f(p(x_1), p(y_1))$. Now we define the subspace of antisymmetric square-integrable functions as $L_A^2(\mathbb{R}^4) = \{\psi | \psi \in L^2(\mathbb{R}^4) \text{ and } p\psi = (-1)^{\pi(p)}\psi\}$, whereby $\pi(p)$ is the parity of the permutation p . By standard results [15, 16] the Hamiltonian H is self-adjoint

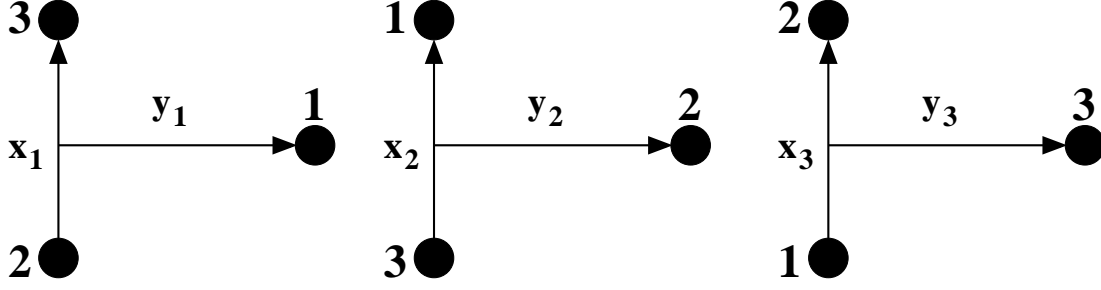


FIG. 1. Three sets of Jacobi coordinates. The picture shows only directions of the vectors, the scales are chosen in order to ensure that $H_0 = -\Delta_{x_k} - \Delta_{y_k}$ for $k = 1, 2, 3$.

on $L_A^2(\mathbb{R}^4)$ with the domain $D(H) = L_A^2(\mathbb{R}^4) \cap \mathcal{H}^2(\mathbb{R}^4)$, where $\mathcal{H}^2(\mathbb{R}^4)$ is the corresponding Sobolev space [16, 17].

The subsystem of 2 fermions is described by the Hamiltonian $h(1)$, where

$$h(\lambda) = -\Delta_x + \lambda v(|x|) \quad (6)$$

and $\lambda > 0$ is a coupling constant. The Hamiltonian (6) acts on the subspace $L_A^2(\mathbb{R}^2)$, where $L_A^2(\mathbb{R}^2) = \{\phi \mid \phi \in L^2(\mathbb{R}^2) \text{ and } \phi(x) = -\phi(-x)\}$. $h(\lambda)$ is self-adjoint on $L_A^2(\mathbb{R}^2)$ with the domain $D(h) = L_A^2(\mathbb{R}^2) \cap \mathcal{H}^2(\mathbb{R}^2)$. We shall say that the interaction $v(x)$ is *tuned to the p-wave zero energy resonance* if $h(1) \geq 0$ and $h(1 + \varepsilon) \not\geq 0$ for all $\varepsilon > 0$.

Let $N_z(H)$ denote the number of bound states of H , whose energy is less than $-z^2$. Our aim in this paper is to prove the following

Theorem 1. *Suppose that the interactions in (2) are tuned to the zero energy p-wave resonance. Then $\lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} N_z(H) = 8/(3\pi)$.*

Remark. Theorem 1 provides a firm mathematical footing for the super Efimov effect. We do not prove it here, but one can show that $\lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} N_z^\pm(H) = 4/(3\pi)$, where $N_z^\pm(H)$ is the number of bound states of H , which have the energy less than $-z^2$ and angular momentum ± 1 respectively. This agrees with the results in [4].

From now on we shall always assume that the interaction of 2 spinless fermions is tuned to the zero energy p-wave resonance. Consider the integral operator on $L_A^2(\mathbb{R}^2)$

$$k(z) := |v|^{\frac{1}{2}}(-\Delta_x + z^2)^{-1}|v|^{\frac{1}{2}}, \quad (7)$$

which is called the Birman-Schwinger (BS) operator. Its integral kernel has the form (eq.

(7.2) in [6])

$$k(x, y) = (2\pi)^{-1} |v(x)|^{\frac{1}{2}} K_0(z|x-y|) |v(y)|^{\frac{1}{2}}, \quad (8)$$

where

$$K_0(t) = -\tilde{I}_0(t) \ln \frac{t}{2} + \sum_{m=0}^{\infty} \frac{t^{2m}}{2^{2m}(m!)^2} \psi(m+1), \quad (9)$$

$$\tilde{I}_0(t) = \sum_{l=1}^{\infty} [(l!) \Gamma(l+1)]^{-1} 2^{-2l} t^{2l}, \quad (10)$$

$$\psi(j) = 1 + \frac{1}{2} + \dots + \frac{1}{j} - 1 - C \quad (11)$$

with C being Euler's constant. Note that contrary to [6] the summation in (10) starts from $l = 1$ because the term produced in $k(z)$ by $l = 0$ is identically zero on $L_A^2(\mathbb{R}^2)$ (this term, which is responsible for the projection operator term in (7.3) in [6], is absent in our case). Thus we have [6] $k(z) = [\sum_{k=0}^{\infty} A_k z^{2k}] z^2 \ln z + [\sum_{k=0}^{\infty} B_k z^{2k}]$, where the series in square brackets sum up to entire analytic operator functions and the coefficients A_k, B_k are Hilbert-Schmidt operators. The operator $k(0)$ is compact and in the vicinity of $z = 0$ the operator $k(z)$ is compact as well. Because the interaction is tuned to the p-wave zero energy resonance from the BS principle (see Theorem 9 in [12]) we infer that $\|k(0)\| = 1$. By standard results in quantum mechanics the ground state of $h(\lambda)$ for $\lambda > 1$ is doubly degenerate with the angular momentum $l = \pm 1$. By the BS principle [12] it follows immediately that $\|k(0)\| = 1$ is an eigenvalue of $k(0)$ with multiplicity 2. Due to spherical symmetry the largest eigenvalue of $k(z)$ for $z > 0$ is also doubly degenerate. By the analysis in [6] in the vicinity of $z = 0$ one has

$$k(z)\varphi_{\pm}(z) = \mu(z)\varphi_{\pm}(z), \quad (12)$$

where $z \geq 0$, $\mu(z) = \sup \sigma(k(z))$, $\|\varphi_{\pm}(z)\| = 1$, $\mu(0) = 1$. The orthogonal eigenvectors $\varphi_{\pm}(z)$ are defined for all $z \geq 0$ and are eigenfunctions of the angular momentum operator with the eigenvalues $l = \pm 1$ respectively. By standard results in perturbation theory we have

$$\varphi_{\pm}(z) = \eta_{\pm} + \mathcal{O}(z^2 \ln z), \quad (13)$$

where $\eta_{\pm} \equiv \varphi_{\pm}(0)$. Due to the spherical symmetry of the potential $\eta_{\pm}(x) = \eta_0(|x|)e^{\pm i\varphi_x}$, where $|x|, \varphi_x$ are polar coordinates. By perturbation theory [6] $\mu(z)$ has a convergent expansion in the vicinity of $z = 0$ given by the series $\mu(z) = \sum_{n,m \geq 0}^{\infty} c_{nm} (z^2 \ln z)^n z^{2m}$, where

$c_{nm} \in \mathbb{R}$. The leading terms of perturbation series are given by the expression

$$\mu(z) = 1 + \frac{\pi}{2} c_0^2 z^2 \ln z + \mathcal{O}(z^2), \quad (14)$$

where

$$\begin{aligned} c_0^2 &= -\frac{1}{4\pi^2} \int \int |v(|x|)|^{\frac{1}{2}} |v(|y|)|^{\frac{1}{2}} |x - y|^2 \eta_+^*(x) \eta_+(y) d^2x d^2y \\ &= \left[\int_0^\infty s^2 \eta_0(s) |v(s)|^{\frac{1}{2}} \right]^2 \end{aligned} \quad (15)$$

(see also the text below eq. (7.13) in [6]). Note that due to $k(z) > k(z')$ for $z' > z > 0$ the function $\mu(z)$ is monotone decreasing on $[0, \infty)$ and

$$\sup_{z>0} \left\| [1 - \varphi_+(z)(\varphi_+(z), \cdot) - \varphi_-(z)(\varphi_-(z), \cdot)] k(z) \right\| < 1. \quad (16)$$

Now we consider the 3-body problem. We denote $v_\alpha := v(|r_\beta - r_\gamma|)$, where (α, β, γ) is any permutation of the numbers $(1, 2, 3)$. Let us introduce the linear subspace $\mathcal{H}_A \subset L^2(\mathbb{R}^4) \oplus L^2(\mathbb{R}^4) \oplus L^2(\mathbb{R}^4)$, where each vector $(\phi_1, \phi_2, \phi_3) \in \mathcal{H}_A$ satisfies the antisymmetry requirements listed in Table I. Each operator p_{ik} in Table I permutes spatial coordinates of particles i, k . Let us consider the operator $M(z)$ on \mathcal{H}_A , whose matrix entries are the following operators

$$M_{\alpha\beta}(z) := |v_\alpha|^{\frac{1}{2}} (H_0 + z^2)^{-1} |v_\beta|^{\frac{1}{2}}. \quad (17)$$

For each set of Jacobi coordinates in Fig. 1 we introduce the Fourier transform \mathcal{F}_k , which acts on $f(x_k, y_k) \in L^1(\mathbb{R}^4)$ as follows

$$\hat{f}(p_k, q_k) := \frac{1}{(2\pi)^2} \int d^2x_k d^2y_k e^{-i(x_k \cdot p_k + y_k \cdot q_k)} f(x_k, y_k). \quad (18)$$

For any interval $\Omega \subset \mathbb{R}$ let us define the cutoff operator on \mathcal{H}_A

$$\mathfrak{X}_\Omega = \text{diag} \{ \mathcal{F}_1^{-1} \chi_\Omega(|q_1|) \mathcal{F}_1, \mathcal{F}_2^{-1} \chi_\Omega(|q_2|) \mathcal{F}_2, \mathcal{F}_3^{-1} \chi_\Omega(|q_3|) \mathcal{F}_3 \}. \quad (19)$$

We separate the diagonal part of $M(z)$ by writing $M(z) = M'(z) + M_d(z)$, where $M_d(z) := \text{diag}\{M_{11}(z), M_{22}(z), M_{33}(z)\}$ and $M'(z) = M(z) - M_d(z)$. The operator $M'(z)$ is compact for all $z > 0$. Indeed, we can write

$$\begin{aligned} M'(z) &= \mathfrak{X}_{[0,R]} M'(z) \mathfrak{X}_{[0,R]} + \mathfrak{X}_{(R,\infty)} M'(z) \mathfrak{X}_{[0,R]} \\ &\quad + \mathfrak{X}_{[0,R]} M'(z) \mathfrak{X}_{(R,\infty)} + \mathfrak{X}_{(R,\infty)} M'(z) \mathfrak{X}_{(R,\infty)}. \end{aligned} \quad (20)$$

Since the interactions are bounded it is easy to see that the norm of each of the last three terms is $o(R)$ for $R \rightarrow \infty$. Hence, the compactness of the lhs of (20) follows from the compactness of the first term on the rhs for all $R > 0$. We prove its compactness by proving the same for each of its matrix entries considered as operators on $L^2(\mathbb{R}^4)$. The operator $\mathcal{F}_1 \mathfrak{X}_{[0,R]} M_{12}(z) \mathfrak{X}_{[0,R]} \mathcal{F}_2^{-1}$ has the kernel

$$\hat{M}_{12}(p_1, q_1; p'_1, q'_1) = \frac{1}{\pi^2} \chi_{[0,R]}(|q_1|) \frac{\widehat{|v|^{\frac{1}{2}}}\left(p_1 + \frac{2}{\sqrt{3}}q'_1 + \frac{1}{\sqrt{3}}q_1\right) \widehat{|v|^{\frac{1}{2}}}\left(\frac{1}{\sqrt{3}}q'_1 + \frac{2}{\sqrt{3}}q_1 - p'_1\right)}{(2q'_1 + q_1)^2 + 3q_1^2 + 3z^2} \chi_{[0,R]}(|q'_1|), \quad (21)$$

where $\widehat{|v|^{\frac{1}{2}}} : \mathbb{R}^2 \rightarrow \mathbb{C}$ is the Fourier transform of $|v(|x|)|^{\frac{1}{2}}$. It is elementary to check that the Hilbert-Schmidt norm of the operator in (21) is finite. First, we prove

Lemma 1. *The following equation holds $N_z(H) = \mathbf{n}(M(z), 1)$.*

Similar lemma has been proved in [3], however, we need to give a new proof in view of antisymmetry restrictions.

Proof of Lemma 1. Consider the operator

$$\mathbf{m}(z) = \sum_{\alpha} (H_0 + z^2)^{-\frac{1}{2}} |v_{\alpha}| (H_0 + z^2)^{-\frac{1}{2}} \quad (22)$$

on the space $L_A^2(\mathbb{R}^4)$. By the BS principle [12] $N_z(H) = \mathbf{n}(\mathbf{m}(z), 1)$. Let L_{λ} and \mathcal{H}_{λ} denote the eigenspaces of the operators $\mathbf{m}(z)$ and $M(z)$ respectively, which correspond to the eigenvalue $\lambda > 1$. Let us first show that the dimension of both eigenspaces is finite. Note that by Theorem 9 in [12] $\sigma_{ess}(\mathbf{m}(z)) \subset (-\infty, 1]$, hence, $\dim L_{\lambda}$ is finite. Due to compactness of $M'(z)$ we have

$$\sigma_{ess}(M(z)) = \sigma_{ess}(M_d(z)) \subseteq [0, \sup_{\alpha} \left\| |v_{\alpha}|^{\frac{1}{2}} (H_0 + z^2)^{-1} |v_{\alpha}|^{\frac{1}{2}} \right\|] \subset [0, 1] \quad (23)$$

and thus $\dim \mathcal{H}_{\lambda}$ is also finite. The operator $B_{\lambda} : L_{\lambda} \rightarrow \mathcal{H}_{\lambda}$ is defined as $(B_{\lambda} \psi)_{\alpha} = |v_{\alpha}|^{1/2} (H_0 + z^2)^{-1/2} \psi$. It is easy to check that this operator is defined correctly, and by applying this operator we infer that from $\dim L_{\lambda} \neq 0$ it follows that $\dim \mathcal{H}_{\lambda} \neq 0$. Similarly, we define the operator $B'_{\lambda} : \mathcal{H}_{\lambda} \rightarrow L_{\lambda}$ given by $B'_{\lambda} \phi = (H_0 + z^2)^{-1/2} \sum_{\beta} |v_{\beta}|^{1/2} \phi_{\beta}$, which is also well-defined. Applying this operator we find that $\dim \mathcal{H}_{\lambda} \neq 0 \iff \dim L_{\lambda} \neq 0$. Since $\lambda^{-1} B'_{\lambda} B_{\lambda} = 1$ we get that $\dim L_{\lambda} = \dim \mathcal{H}_{\lambda}$. Therefore, $N_z(H) = \mathbf{n}(\mathbf{m}(z), 1) = \mathbf{n}(M(z), 1)$. \square

By Lemma 1 and the BS principle [12]

$$N_z(H) = \mathbf{n}(\mathcal{A}(z), 1), \quad (24)$$

where

$$\mathcal{A}(z) = (1 - M_d(z))^{-\frac{1}{2}} M'(z) (1 - M_d(z))^{-\frac{1}{2}}. \quad (25)$$

For $k = 1, 2, 3$ let us introduce the projection operators $P_\pm^{(k)}$ on \mathcal{H}_A , which act on $f(p_k, q_k)$ as follows

$$[P_\pm^{(k)} f](p_k, q_k) = \hat{\eta}_\pm(p_k) \int \hat{\eta}_\pm^*(p'_k) f(p'_k, q_k) dp'_k, \quad (26)$$

and

$$P_\pm = \text{diag} \left\{ \mathcal{F}_1^{-1} P_\pm^{(1)} \mathcal{F}_1, \mathcal{F}_2^{-1} P_\pm^{(2)} \mathcal{F}_2, \mathcal{F}_3^{-1} P_\pm^{(3)} \mathcal{F}_3 \right\}. \quad (27)$$

In (26) $\hat{\eta}_\pm$ are Fourier transformed functions η_\pm defined in (13). Let us fix the cut off parameter $r_\varepsilon \in (0, 1/4)$ and define

$$\mathcal{A}_0(z) = [P_+ + P_-] G(z) M'(z) G(z) [P_+ + P_-], \quad (28)$$

where

$$G(z) = \text{diag} \left\{ \mathcal{F}_1^{-1} g_z(|q_1|) \mathcal{F}_1, \mathcal{F}_2^{-1} g_z(|q_2|) \mathcal{F}_2, \mathcal{F}_3^{-1} g_z(|q_3|) \mathcal{F}_3 \right\}, \quad (29)$$

and $g_z : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined through

$$g_z(s) := \begin{cases} \left(1 - \mu\left(\sqrt{s^2 + z^2}\right)\right)^{-1/2} & \text{if } s \leq r_\varepsilon, \\ 0 & \text{if } s > r_\varepsilon. \end{cases} \quad (30)$$

By (14) there exist $\delta, \delta' > 0$ such that

$$\frac{\delta'}{(s^2 + z^2) |\ln(s^2 + z^2)|} \leq g_z^2(s) \leq \frac{\delta}{(s^2 + z^2) |\ln(s^2 + z^2)|} \quad (31)$$

for $z, s \in (0, r_\varepsilon]$. Besides for $r_\varepsilon \rightarrow 0$ we have $\delta = 4/(\pi c_0^2) + o(r_\varepsilon)$ and $\delta' = 4/(\pi c_0^2) + o(r_\varepsilon)$. We shall always implicitly assume that $z \in (0, r_\varepsilon]$ if not stated otherwise. We decompose the operator $\mathcal{A}(z)$ into a sum of the main term $\mathcal{A}_0(z)$ and the remainder

$$\mathcal{A}(z) = \mathcal{A}_0(z) + \mathcal{R}(z). \quad (32)$$

The operators on the rhs of (32) are self-adjoint and compact. The proof of Theorem 1 is based on the following two theorems

Theorem 2. For all $r_\varepsilon \in (0, 1/4)$ and $a > 0$

$$\lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathbf{n}(\mathcal{A}_0(z), a) = \frac{8}{3\pi a}. \quad (33)$$

Theorem 3. For each $\varepsilon > 0$ one can choose $r_\varepsilon \in (0, 1/4)$ so that

$$\overline{\lim}_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathbf{n}_\mu(\mathcal{R}(z), \varepsilon) < \varepsilon. \quad (34)$$

Sec. III is devoted to the proof of Theorem 2. The proof of Theorem 3, which is rather involved and uses the machinery of trace ideals [13], is given in Sec. IV. Using Theorems 2, 3 we can prove the main theorem

Proof of Theorem 1. For any given $\varepsilon \in (0, 1)$ let us fix $r_\varepsilon > 0$ according to Theorem 3. The eigenvalue distribution function satisfies the inequality [3, 14]

$$\mathbf{n}(A_1 + A_2, a_1 + a_2) \leq \mathbf{n}(A_1, a_1) + \mathbf{n}(A_2, a_2), \quad (35)$$

where $A_{1,2} \in \mathcal{C}(\mathcal{H})$ and $a_{1,2} > 0$. Using this inequality we obtain from (32)

$$\mathbf{n}(\mathcal{A}(z), 1) \leq \mathbf{n}(\mathcal{A}_0(z), 1 - \varepsilon) + \mathbf{n}(\mathcal{R}(z), \varepsilon) \leq \mathbf{n}(\mathcal{A}_0(z), 1 - \varepsilon) + \mathbf{n}_\mu(\mathcal{R}(z), \varepsilon) \quad (36)$$

$$\mathbf{n}(\mathcal{A}_0(z), 1 + \varepsilon) \leq \mathbf{n}(\mathcal{A}(z), 1) + \mathbf{n}(-\mathcal{R}(z), \varepsilon) \leq \mathbf{n}(\mathcal{A}(z), 1) + \mathbf{n}_\mu(\mathcal{R}(z), \varepsilon). \quad (37)$$

(Because $\mathcal{R}(z)$ is self-adjoint we have $\mathbf{n}(\pm \mathcal{R}(z), \varepsilon) \leq \mathbf{n}_\mu(\mathcal{R}(z), \varepsilon)$). Hence, using Theorems 2, 3 we get

$$\begin{aligned} \frac{8}{3\pi(1 + \varepsilon)} - \varepsilon &\leq \underline{\lim}_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathbf{n}(\mathcal{A}(z), 1) \\ &\leq \overline{\lim}_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathbf{n}(\mathcal{A}(z), 1) \leq \frac{8}{3\pi(1 - \varepsilon)} + \varepsilon. \end{aligned} \quad (38)$$

Letting $\varepsilon \rightarrow 0$ and using (24) we complete the proof. \square

III. SPECTRAL ASYMPTOTIC OF THE LEADING TERM

Definition 1. For an operator function $B : \mathbb{R}_+ / \{0\} \rightarrow \mathcal{C}(\mathcal{H})$ we shall write $B(z) = \mathcal{O}_C(z)$ if and only if for all $\epsilon > 0$ there exists $z_0 > 0$ and a decomposition $B(z) = B_\epsilon(z) + P_\epsilon(z)$, where $B_\epsilon, P_\epsilon : \mathbb{R}_+ / \{0\} \rightarrow \mathcal{C}(\mathcal{H})$ are such that $\sup_{z \in (0, z_0)} \|B_\epsilon(z)\| < \epsilon$ and $\sup_{z \in (0, z_0)} \dim \text{Ran} P_\epsilon(z) < \infty$.

TABLE I. Antisymmetry relations for $(\phi_1, \phi_2, \phi_3) \in \mathcal{H}_A$.

$p_{12}\phi_1 = -\phi_2$	$p_{12}\phi_2 = -\phi_1$	$p_{12}\phi_3 = -\phi_3$
$p_{13}\phi_1 = -\phi_3$	$p_{13}\phi_2 = -\phi_2$	$p_{13}\phi_3 = -\phi_1$
$p_{13}\phi_1 = -\phi_1$	$p_{13}\phi_2 = -\phi_3$	$p_{23}\phi_3 = -\phi_2$

Remark. If $B(z) = \mathcal{O}_C(z)$ in Def. 1 is such that $B(z) = B^*(z)$ then in the decomposition one can choose B_ϵ, P_ϵ so that $B_\epsilon(z) = B_\epsilon^*(z)$ and $P_\epsilon(z) = P_\epsilon^*(z)$. (This can be verified by writing $B(z) = (1/2)[B_\epsilon(z) + B_\epsilon^*(z)] + (1/2)[P_\epsilon(z) + P_\epsilon^*(z)]$).

The following proposition is obvious

Proposition 1. *Suppose that $B : \mathbb{R}_+/\{0\} \rightarrow \mathcal{C}(\mathcal{H})$ is such that $\sup_{z \in (0, z_0)} \|B(z)\|_{HS} < \infty$ for some $z_0 > 0$. Then $B(z) = \mathcal{O}_C(z)$.*

Proof. Let us write the singular value decomposition [13]

$$B(z) = \sum_{k=1}^{\infty} \mu_k(B(z)) \left(\phi_k(z), \cdot \right) \psi_k(z), \quad (39)$$

where $\{\phi_k(z)\}, \{\psi_k(z)\}$ are orthonormal sets. For $z \in (0, z_0)$ the following inequality holds

$$n\mu_n^2(z) \leq \mu_1^2(z) + \dots + \mu_n^2(z) \leq \|B(z)\|_{HS}^2 \leq \alpha, \quad (40)$$

where $\alpha := \sup_{z \in (0, z_0)} \|B(z)\|_{HS}$. For any given $\epsilon > 0$ we can set n equal to the integer, which is larger than $\alpha\epsilon^{-2}$. Then $B_\epsilon(z) := \sum_{k=n+1}^{\infty} \mu_k(z) (\phi_k(z), \cdot) \psi_k(z)$ and $P_\epsilon(z) := \sum_{k=1}^n \mu_k(z) (\phi_k(z), \cdot) \psi_k(z)$ fulfill the requirement in Def. 1. \square

Definition 2. *Consider two operator functions $A : \mathbb{R}_+/\{0\} \rightarrow \mathcal{C}(\mathcal{H}_1)$, $B : \mathbb{R}_+/\{0\} \rightarrow \mathcal{C}(\mathcal{H}_2)$ such that $A^*(z) = A(z)$, $B^*(z) = B(z)$. We shall say that $A(z)$ and $B(z)$ are equivalent and write $A(z) \sim B(z)$ if either $\lambda_n(A(z)) = \lambda_n(B(z))$ for $n = 1, 2, \dots$ or $A(z) - B(z) = \mathcal{O}_C(z)$ (the last case implies $\mathcal{H}_1 = \mathcal{H}_2$).*

Let us explain the point of Definition 2. Below we shall prove that $\mathcal{A}_0(z)$ is equivalent to some operator function $T(z)$, whose spectrum is known explicitly. Then we shall prove that $\lim_{z \rightarrow 0} |\ln |\ln z||^{-1} \mathbf{n}(\mathcal{A}_0(z), a) = \lim_{z \rightarrow 0} |\ln |\ln z||^{-1} \mathbf{n}(T(z), a)$ for $a > 0$ thus proving Theorem 1.

Let us analyze the spectrum of $\mathcal{A}_0(z)$. If $\mathcal{A}_0(z)\Psi = \lambda\Psi$, where $\Psi \in \mathcal{H}_A$ and $\lambda \neq 0$ then due to antisymmetry requirements listed in Table I we have

$$\Psi = \begin{pmatrix} \mathcal{F}_1^{-1}[f_+(q_1)\hat{\eta}_+(p_1) + f_-(q_1)\hat{\eta}_-(p_1)] \\ \mathcal{F}_2^{-1}[f_+(q_2)\hat{\eta}_+(p_2) + f_-(q_2)\hat{\eta}_-(p_2)] \\ \mathcal{F}_3^{-1}[f_+(q_3)\hat{\eta}_+(p_3) + f_-(q_3)\hat{\eta}_-(p_3)] \end{pmatrix}. \quad (41)$$

Substituting the last ansatz into the equation $\mathcal{A}_0(z)\Psi = \lambda\Psi$ and using (5) we find that f_{\pm} satisfy integral equation

$$L^{(a)}(z) \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} L_{11}^{(a)}(z) & L_{12}^{(a)}(z) \\ L_{21}^{(a)}(z) & L_{22}^{(a)}(z) \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \lambda \begin{pmatrix} f_+ \\ f_- \end{pmatrix}. \quad (42)$$

The matrix entries $L_{ik}^{(a)}(z)$ are integral operators on $L^2(\mathbb{R}^2)$ with the kernels

$$L_{11}^{(a)}(p, q) = \frac{-2\hat{\psi}_+^* \left(\frac{2}{\sqrt{3}}q + \frac{1}{\sqrt{3}}p \right) \hat{\psi}_+ \left(\frac{2}{\sqrt{3}}p + \frac{1}{\sqrt{3}}q \right) g_z(|p|)g_z(|q|)}{(p^2 + q^2 + p \cdot q + z^2)}, \quad (43)$$

$$L_{12}^{(a)}(p, q) = \frac{-2\hat{\psi}_+^* \left(\frac{2}{\sqrt{3}}q + \frac{1}{\sqrt{3}}p \right) \hat{\psi}_- \left(\frac{2}{\sqrt{3}}p + \frac{1}{\sqrt{3}}q \right) g_z(|p|)g_z(|q|)}{(p^2 + q^2 + p \cdot q + z^2)}, \quad (44)$$

$$L_{22}^{(a)}(p, q) = \frac{-2\hat{\psi}_-^* \left(\frac{2}{\sqrt{3}}q + \frac{1}{\sqrt{3}}p \right) \hat{\psi}_- \left(\frac{2}{\sqrt{3}}p + \frac{1}{\sqrt{3}}q \right) g_z(|p|)g_z(|q|)}{(p^2 + q^2 + p \cdot q + z^2)}, \quad (45)$$

and $L_{21}^{(a)}(p, q) = (L_{12}^{(a)}(q, p))^*$. In (43)-(45) $\hat{\psi}_{\pm}(p)$ are Fourier transforms of the functions

$$\psi_{\pm}(x) := |v(|x|)|^{\frac{1}{2}}\eta_{\pm}(x). \quad (46)$$

The operator function $L^{(a)}(z)$ acts on $L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ and clearly $L^{(a)}(z) \sim \mathcal{A}_0(z)$. The relevant properties of the functions $\hat{\psi}_{\pm}(p)$ are summarized in the following

Lemma 2. *In polar coordinates $\hat{\psi}_{\pm}(p) = \psi_0(|p|)e^{\pm i\varphi_p}$, where $\psi_0 \in L^2((0, \infty); xdx)$. There are $\alpha, \beta, \gamma > 0$ such that*

$$|\hat{\psi}_{\pm}(p)| \leq \alpha|p|, \quad (47)$$

$$|\hat{\psi}_{\pm}(p+q) - \hat{\psi}_{\pm}(p) - \hat{\psi}_{\pm}(q)| \leq \beta|p||q|, \quad (48)$$

$$s^{-2}\psi_0^* \left(\frac{1}{\sqrt{3}}s \right) \psi_0 \left(\frac{2}{\sqrt{3}}s \right) = \frac{c_0^2}{6} + \omega(s), \quad (49)$$

where $|\omega(s)| \leq \gamma s^2$ and c_0^2 is defined in (15).

Proof. The representation in polar coordinates follows immediately from symmetry arguments. The Fourier transformed function is

$$\hat{\psi}_{\pm}(p) := \frac{1}{2\pi} \int e^{-ip \cdot r} \psi_{\pm}(r) d^2r = \frac{1}{2\pi} \int (e^{-ip \cdot r} - 1) \psi_{\pm}(r) d^2r. \quad (50)$$

Using that $|e^{-ip \cdot r} - 1| \leq 2$ we obtain

$$|\hat{\psi}_{\pm}(p)| \leq |p| \pi^{-1} \left\| |r| |v(|r|)|^{\frac{1}{2}} \eta_{\pm}(r) \right\|_1 \leq |p| \pi^{-1} \left\| |r| |v(|r|)|^{\frac{1}{2}} \right\|_2. \quad (51)$$

(48) is obtained similarly using the inequality

$$|e^{-i(p+q) \cdot r} - (e^{-ip \cdot r} - 1) - e^{-iq \cdot r}| = |(e^{-iq \cdot r} - 1)(e^{-ip \cdot r} - 1)| \leq 4|p||q|r^2. \quad (52)$$

Expanding the exponent in the Fourier transform we obtain the expression

$$\hat{\psi}_{\pm}(p) = \frac{-i}{2} |p| e^{\pm i \varphi_p} \int_0^{\infty} s^2 \eta_0(s) |v(s)|^{\frac{1}{2}} + \mathcal{O}(|p|^3), \quad (53)$$

from which (49) follows. \square

Using Lemmas 2, 4 we conclude that $L^{(a)}(z) \sim L^{(b)}(z)$, where the matrix entries of $L^{(b)}(z)$ are integral operators with the kernels

$$L_{11}^{(b)}(p, q) = \frac{-2 \left[\hat{\psi}_+^* \left(\frac{1}{\sqrt{3}} p \right) \hat{\psi}_+ \left(\frac{2}{\sqrt{3}} p \right) + \hat{\psi}_+^* \left(\frac{2}{\sqrt{3}} q \right) \hat{\psi}_+ \left(\frac{1}{\sqrt{3}} q \right) \right] g_z(|p|) g_z(|q|)}{(p^2 + q^2 + p \cdot q + z^2)}, \quad (54)$$

$$L_{12}^{(b)}(p, q) = \frac{-2 \left[\hat{\psi}_+^* \left(\frac{1}{\sqrt{3}} p \right) \hat{\psi}_- \left(\frac{2}{\sqrt{3}} p \right) + \hat{\psi}_+^* \left(\frac{2}{\sqrt{3}} q \right) \hat{\psi}_- \left(\frac{1}{\sqrt{3}} q \right) \right] g_z(|p|) g_z(|q|)}{(p^2 + q^2 + p \cdot q + z^2)}, \quad (55)$$

$$L_{22}^{(b)}(p, q) = \frac{-2 \left[\hat{\psi}_-^* \left(\frac{1}{\sqrt{3}} p \right) \hat{\psi}_- \left(\frac{2}{\sqrt{3}} p \right) + \hat{\psi}_-^* \left(\frac{2}{\sqrt{3}} q \right) \hat{\psi}_- \left(\frac{1}{\sqrt{3}} q \right) \right] g_z(|p|) g_z(|q|)}{(p^2 + q^2 + p \cdot q + z^2)}, \quad (56)$$

and $L_{21}^{(b)}(p, q) = (L_{12}^{(b)}(q, p))^*$. The operator $L^{(b)}(z)$ acts on $L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$. By (47) we have

$$\begin{aligned} & \left| \hat{\psi}_+^* \left((1/\sqrt{3})p \right) \hat{\psi}_+ \left((2/\sqrt{3})p \right) \right| \left| \frac{1}{p^2 + q^2 + p \cdot q + z^2} - \frac{1}{p^2 + q^2} \right| \\ & \leq \frac{2\alpha^2}{3} \frac{|p||q|}{p^2 + q^2 + p \cdot q} + \frac{2\alpha^2}{3} \frac{z^2}{p^2 + q^2 + p \cdot q + z^2}. \end{aligned} \quad (57)$$

Hence, by Lemma 4 we have $L^{(b)}(z) \sim L^{(c)}(z)$, where $L^{(c)}(z)$ has the matrix entries

$$L_{11}^{(c)}(p, q) = \frac{-2 \left[\hat{\psi}_+^* \left(\frac{1}{\sqrt{3}}p \right) \hat{\psi}_+ \left(\frac{2}{\sqrt{3}}p \right) + \hat{\psi}_+^* \left(\frac{2}{\sqrt{3}}q \right) \hat{\psi}_+ \left(\frac{1}{\sqrt{3}}q \right) \right] g_z(|p|)g_z(|q|)}{(p^2 + q^2)}, \quad (58)$$

$$L_{12}^{(c)}(p, q) = \frac{-2 \left[\hat{\psi}_+^* \left(\frac{1}{\sqrt{3}}p \right) \hat{\psi}_- \left(\frac{2}{\sqrt{3}}p \right) + \hat{\psi}_+^* \left(\frac{2}{\sqrt{3}}q \right) \hat{\psi}_- \left(\frac{1}{\sqrt{3}}q \right) \right] g_z(|p|)g_z(|q|)}{(p^2 + q^2)}, \quad (59)$$

$$L_{22}^{(c)}(p, q) = \frac{-2 \left[\hat{\psi}_-^* \left(\frac{1}{\sqrt{3}}p \right) \hat{\psi}_- \left(\frac{2}{\sqrt{3}}p \right) + \hat{\psi}_-^* \left(\frac{2}{\sqrt{3}}q \right) \hat{\psi}_- \left(\frac{1}{\sqrt{3}}q \right) \right] g_z(|p|)g_z(|q|)}{(p^2 + q^2)}, \quad (60)$$

and $L_{21}^{(c)}(p, q) = (L_{12}^{(c)}(q, p))^*$. We are interested in the nontrivial spectrum of the compact operator $L^{(c)}(z)$ on $L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, that is we look for solutions of the equation

$$L^{(c)}(z) \begin{pmatrix} F_+ \\ F_- \end{pmatrix} = \begin{pmatrix} L_{11}^{(c)}(z) & L_{12}^{(c)}(z) \\ L_{21}^{(c)}(z) & L_{22}^{(c)}(z) \end{pmatrix} \begin{pmatrix} F_+ \\ F_- \end{pmatrix} = \lambda \begin{pmatrix} F_+ \\ F_- \end{pmatrix}, \quad (61)$$

where $\lambda \neq 0$. Now we employ the symmetry of integral equations and expand F_+, F_- as follows

$$F_+(p) = \sum_{l=-\infty}^{\infty} e^{i(l-1)\varphi_p} f_+^{(l)}(|p|), \quad (62)$$

$$F_-(p) = \sum_{l=-\infty}^{\infty} e^{i(l+1)\varphi_p} f_-^{(l)}(|p|), \quad (63)$$

where $f_{\pm}^{(l)}(x) \in L^2([0, r_\varepsilon]; xdx)$. Substituting (62), (63) into (61) we find that $f_+^{(l)}(|p|) = 0$ for all l except $l = \pm 1$. Thus we conclude that $L^{(c)}(z) \sim T^+(z) \oplus T^-(z)$, where $T^\pm(z)$ act on $L^2([0, r_\varepsilon]; xdx) \oplus L^2([0, r_\varepsilon]; xdx)$ and have the structure

$$T^\pm(z) = \begin{pmatrix} T_{11}^\pm(z) & T_{12}^\pm(z) \\ T_{21}^\pm(z) & T_{22}^\pm(z) \end{pmatrix}. \quad (64)$$

The matrix entries are integral operators on $L^2([0, r_\varepsilon]; xdx)$ with the following kernels

$$T_{11}^+(s, t) = T_{22}^-(s, t) = (-4\pi) \frac{\left[\psi_0^* \left(\frac{1}{\sqrt{3}}s \right) \psi_0 \left(\frac{2}{\sqrt{3}}s \right) + \psi_0^* \left(\frac{2}{\sqrt{3}}t \right) \psi_0 \left(\frac{1}{\sqrt{3}}t \right) \right] g_z(s)g_z(t)}{(s^2 + t^2)}, \quad (65)$$

$$T_{12}^+(s, t) = T_{21}^-(s, t) = (-4\pi) \frac{\psi_0 \left(\frac{1}{\sqrt{3}}t \right) \psi_0^* \left(\frac{2}{\sqrt{3}}t \right) g_z(s)g_z(t)}{(s^2 + t^2)}, \quad (66)$$

$$T_{21}^+(s, t) = T_{12}^-(s, t) = (-4\pi) \frac{\psi_0^* \left(\frac{1}{\sqrt{3}}s \right) \psi_0 \left(\frac{2}{\sqrt{3}}s \right) g_z(s)g_z(t)}{(s^2 + t^2)}, \quad (67)$$

$$T_{22}^+(s, t) = T_{11}^-(s, t) = 0. \quad (68)$$

Below we shall consider only the spectrum of $T^+(z)$. The spectrum of $T^-(z)$ is considered analogously. Let us introduce two integral operator functions $B_{1,2}(z)$ on $L^2([0, r_\varepsilon]; xdx)$ with the following integral kernels

$$B_1(s, t) = \frac{\psi_0^* \left(\frac{1}{\sqrt{3}}s \right) \psi_0 \left(\frac{2}{\sqrt{3}}s \right) g_z(s)g_z(t) \chi_{\{s < t\}}}{(s^2 + t^2)}, \quad (69)$$

$$B_2(s, t) = \psi_0^* \left(\frac{1}{\sqrt{3}}s \right) \psi_0 \left(\frac{2}{\sqrt{3}}s \right) g_z(s)g_z(t) \chi_{\{s \geq t\}} \left[\frac{1}{s^2 + t^2} - \frac{1}{s^2} \right]. \quad (70)$$

Our aim is to prove that $B_{1,2}(z) = \mathcal{O}_C(z)$. The function $\chi_{\{s < t\}}$ is such that $\chi_{\{s < t\}} = 1$ if $s < t$ and $\chi_{\{s < t\}} = 0$ if $s \geq t$ (the notation for this function using other relation symbols is self-explanatory). Using Lemma 2 and (31) for $z \in (0, r_\varepsilon]$ we get

$$\begin{aligned} \|B_1(z)\|_{HS}^2 &\leq \delta^2 \alpha^4 \int_0^{r_\varepsilon} \int_0^{r_\varepsilon} \frac{s^5 t \chi_{\{s < t\}} ds dt}{(s^2 + z^2) |\ln(s^2 + z^2)| (t^2 + z^2) |\ln(t^2 + z^2)| (s^2 + t^2)^2} \\ &\leq \delta^2 \alpha^4 \int_0^{r_\varepsilon} \int_0^{r_\varepsilon} \frac{s^2 ds dt}{|\ln s^2| |\ln t^2| (s^2 + t^2)^2}, \end{aligned} \quad (71)$$

where we have used that for small $s, z \in (0, r_\varepsilon]$ one has $(s^2 + z^2) |\ln(s^2 + z^2)| \geq s^2 |\ln s^2|$. In the last integral we pass to polar coordinates $s = \rho \sin \phi$, $t = \rho \cos \phi$ and obtain the inequality

$$\|B_1(z)\|_{HS}^2 \leq \frac{\delta^2 \alpha^4}{4} \int_0^{2r_\varepsilon} \int_0^{\pi/2} \frac{\sin^2 \phi \, d\rho \, d\phi}{\rho |\ln \rho|^2} = \frac{\delta^2 \alpha^4 \pi}{16} \int_0^{2r_\varepsilon} \frac{d\rho}{\rho |\ln \rho|^2}. \quad (72)$$

The last integral converges [19] and from Proposition 1 it follows that $B_1(z) = \mathcal{O}_C(z)$. Similarly one shows that $B_2(z) = \mathcal{O}_C(z)$. Using this fact we obtain $T^+(z) \sim T^{(a)}(z)$, where $T^{(a)}(z)$ acts on the same space as $T^+(z)$ and its matrix entries have the following integral kernels

$$\begin{aligned} T_{11}^{(a)}(s, t) &= (-4\pi) \left[s^{-2} \psi_0^* \left(\frac{1}{\sqrt{3}}s \right) \psi_0 \left(\frac{2}{\sqrt{3}}s \right) \chi_{\{s \geq t\}} \right. \\ &\quad \left. + t^{-2} \psi_0^* \left(\frac{2}{\sqrt{3}}t \right) \psi_0 \left(\frac{1}{\sqrt{3}}t \right) \chi_{\{s \leq t\}} \right] g_z(s)g_z(t), \end{aligned} \quad (73)$$

$$T_{12}^{(a)}(s, t) = (-4\pi) t^{-2} \psi_0 \left(\frac{1}{\sqrt{3}}t \right) \psi_0^* \left(\frac{2}{\sqrt{3}}t \right) \chi_{\{s \leq t\}} g_z(s)g_z(t), \quad (74)$$

$$T_{21}^{(a)}(s, t) = (-4\pi) s^{-2} \psi_0^* \left(\frac{1}{\sqrt{3}}s \right) \psi_0 \left(\frac{2}{\sqrt{3}}s \right) \chi_{\{s \geq t\}} g_z(s)g_z(t), \quad (75)$$

$T_{22}^{(a)}(s, t) = 0$. Now let us consider the expression in square brackets in (73). Due to (49) we

have a. e.

$$\begin{aligned} & s^{-2}\psi_0^* \left(\frac{1}{\sqrt{3}}s \right) \psi_0 \left(\frac{2}{\sqrt{3}}s \right) \chi_{\{s \geq t\}} + t^{-2}\psi_0^* \left(\frac{2}{\sqrt{3}}t \right) \psi_0 \left(\frac{1}{\sqrt{3}}t \right) \chi_{\{s \leq t\}} \\ &= \frac{c_0^2}{6} + \omega(s)\chi_{\{s \geq t\}} + \chi_{\{s \leq t\}}\omega(t) = \frac{c_0^2}{6} + \omega(s) + \omega(t) - [\omega(s)\chi_{\{s \leq t\}} + \chi_{\{s \geq t\}}\omega(t)]. \end{aligned} \quad (76)$$

Using (31) and (49) one can easily check that $B_3(z) = \mathcal{O}_C(z)$, where the integral operator $B_3(z)$ acts on $L^2([0, r_\varepsilon]; xdx)$ and has the kernel

$$B_3(s, t) = \omega(s)\chi_{\{s \leq t\}}g_z(s)g_z(t). \quad (77)$$

Making similar decompositions for other kernels in (73)-(75) and using that $B_3(z) = \mathcal{O}_C(z)$ we conclude that

$$T^{(a)}(z) \sim S(z) - \frac{2\pi c_0^2}{3}\mathcal{T}(z), \quad (78)$$

where $S(z), \mathcal{T}(z)$ act on the same space as $T^{(a)}(z)$ and their matrix entries have the following integral kernels

$$S_{11}(s, t) = (-4\pi) \left[\frac{c_0^2}{6} + \omega(s) + \omega(t) \right] g_z(s)g_z(t) \quad (79)$$

$$S_{12}(s, t) = (-4\pi)\omega(t)g_z(s)g_z(t), \quad (80)$$

$$S_{21}(s, t) = (-4\pi)\omega(s)g_z(s)g_z(t), \quad (81)$$

and

$$\mathcal{T}_{12}(s, t) = \chi_{\{s \leq t\}}g_z(s)g_z(t), \quad (82)$$

$$\mathcal{T}_{21}(s, t) = \chi_{\{s \geq t\}}g_z(s)g_z(t), \quad (83)$$

$$\mathcal{T}_{11}(s, t) = \mathcal{T}_{22}(s, t) = S_{22}(s, t) = 0. \quad (84)$$

Finally, from (78) we conclude that

$$T^+(z) \sim -\frac{2\pi c_0^2}{3}\mathcal{T}(z) \quad (85)$$

because $S(z)$ for all $z > 0$ is a rank 3 operator. The nonzero spectrum of the operator $\mathcal{T}(z)$ can be calculated explicitly. Note that $\sigma(\mathcal{T}(z))/\{0\} = \sigma(\mathcal{T}'(z))/\{0\}$, where

$$\mathcal{T}'_{12}(s, t) = \chi_{\{s \leq t\}}g_z^2(t), \quad (86)$$

$$\mathcal{T}'_{21}(s, t) = \chi_{\{s \geq t\}}g_z^2(t), \quad (87)$$

$$\mathcal{T}'_{11}(s, t) = \mathcal{T}'_{22}(s, t) = 0. \quad (88)$$

(This is the consequence of the fact that $\sigma(AB)/\{0\} = \sigma(BA)/\{0\}$ for any bounded A, B , see [20]). The equation $\mathcal{T}'(z)f = \lambda f$ for $\lambda \neq 0$ takes the form

$$\lambda f_1(s) = \int_s^{r_\varepsilon} f_2(t) g_z^2(t) t dt \quad (89)$$

$$\lambda f_2(s) = \int_0^s f_1(t) g_z^2(t) t dt. \quad (90)$$

Let us make the change of variables in (89)-(90) setting $x = \xi(s)$, where

$$\xi(s) := \int_0^s g_z^2(t) t dt \quad (91)$$

is monotone increasing. Then Eqs. (89)-(90) take the form

$$\lambda \tilde{f}_1(x) = \int_x^{\xi(r_\varepsilon)} \tilde{f}_2(x') dx' \quad (92)$$

$$\lambda \tilde{f}_2(x) = \int_0^x \tilde{f}_1(x') dx', \quad (93)$$

where $\tilde{f}_i \in C^1([0, \xi(r_\varepsilon)])$. Similar integral equations were obtained in [4, 21]. Differentiating (92)-(93) over x gives $\lambda(d\tilde{f}_1/dx) = -\tilde{f}_2(x)$; $\lambda(d\tilde{f}_2/dx) = \tilde{f}_1(x)$. These differential equations are solved by $\tilde{f}_1(x) = \cos(\lambda^{-1}x + \varphi_\lambda)$, and $\tilde{f}_2(x) = \sin(\lambda^{-1}x + \varphi_\lambda)$. Substituting these expressions back into (92)-(93) we find that $\mathcal{T}(z)$ has an infinite number of positive and negative eigenvalues, namely,

$$\lambda_k(\mathcal{T}(z)) = \lambda_k(-\mathcal{T}(z)) = \frac{\xi(r_\varepsilon)}{(\pi/2) + \pi(k-1)}, \quad \text{where } k = 1, 2, \dots \quad (94)$$

Note that

$$\lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \xi(r_\varepsilon) = \frac{2}{\pi c_0^2}, \quad (95)$$

where c_0^2 is defined in (15). Indeed, for any $\rho \in (0, r_\varepsilon)$

$$\lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} [\xi(r_\varepsilon) - \xi(\rho)] = 0. \quad (96)$$

Due to (31) and (91) we obtain

$$\frac{\delta'}{2} \leq \lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \xi(\rho) \leq \frac{\delta}{2}, \quad (97)$$

where we have used that

$$\lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \int_0^\rho \frac{t dt}{(t^2 + z^2) |\ln(t^2 + z^2)|} = \frac{1}{2}. \quad (98)$$

Recall that for $\rho \rightarrow 0$ we have $\delta = 4/(\pi c_0^2) + o(\rho)$ and $\delta' = 4/(\pi c_0^2) + o(\rho)$ (see the text below Eq. (31)), which results in (95). Now let us prove

Lemma 3. Suppose $K : \mathbb{R}_+/\{0\} \rightarrow \mathcal{C}(\mathcal{H})$ is such that $K(z) \sim -\mathcal{T}(z)$. Then for any $a > 0$

$$\lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathbf{n}(K(z), a) = \frac{2}{\pi^2 c_0^2 a}. \quad (99)$$

Proof. From (95) it follows that $\xi(r_\varepsilon) \rightarrow +\infty$ when $z \rightarrow 0$. Hence, from (94) we get

$$\lim_{z \rightarrow 0} [\xi(r_\varepsilon)]^{-1} \mathbf{n}(-\mathcal{T}(z), a) = \frac{1}{\pi a}. \quad (100)$$

We only have to consider the case when $\mathcal{H} = L^2([0, r_\varepsilon]; xdx) \oplus L^2([0, r_\varepsilon]; xdx)$, otherwise the statement is obvious. For any fixed $\epsilon \in (0, a/2)$ there exist $z_0, k > 0$, and self-adjoint $B_\epsilon, P_\epsilon : \mathbb{R}_+ \rightarrow \mathcal{C}(\mathcal{H})$ such that $K(z) = -\mathcal{T}(z) + B_\epsilon(z) + P_\epsilon(z)$, whereby $\sup_{z \in z_0} \|B_\epsilon(z)\| < \epsilon$, and $\sup_{z \in (0, z_0)} \dim \text{Ran} P_\epsilon(z) < k$. By (35)

$$\mathbf{n}(-\mathcal{T}(z), a + 2\epsilon) \leq \mathbf{n}(K(z), a) + \mathbf{n}(-B_\epsilon(z), \epsilon) + \mathbf{n}(-P_\epsilon(z), \epsilon) \quad (101)$$

$$\mathbf{n}(K(z), a) \leq \mathbf{n}(-\mathcal{T}(z), a - 2\epsilon) + \mathbf{n}(B_\epsilon(z), \epsilon) + \mathbf{n}(P_\epsilon(z), \epsilon). \quad (102)$$

Since $\mathbf{n}(\pm B_\epsilon(z), \epsilon) = 0$ and $\mathbf{n}(P_\epsilon(z), \epsilon) \leq k$ we obtain from (101), (102) and (100)

$$\frac{1}{\pi(a + 2\epsilon)} \leq \liminf_{z \rightarrow 0} [\xi(r_\varepsilon)]^{-1} \mathbf{n}(K(z), a) \leq \overline{\lim}_{z \rightarrow 0} [\xi(r_\varepsilon)]^{-1} \mathbf{n}(K(z), a) \leq \frac{1}{\pi(a - 2\epsilon)}. \quad (103)$$

Letting $\epsilon \rightarrow 0$ we prove that

$$\lim_{z \rightarrow 0} [\xi(r_\varepsilon)]^{-1} \mathbf{n}(K(z), a) = \frac{1}{\pi a}. \quad (104)$$

Now the result follows from (95). \square

Proof of Theorem 2. From Lemma 3 and (85) it follows that

$$\lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathbf{n}(T^\pm(z), a) = \lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathbf{n}\left(\frac{3}{2\pi c_0^2} T^\pm(z), \frac{3a}{2\pi c_0^2}\right) = \frac{4}{3\pi a} \quad (105)$$

(we have proved (105) for $T^+(z)$, but the analysis of the operator $T^-(z)$ leads to the same result). By the above analysis $\mathcal{A}_0(z) \sim T^+(z) \oplus T^-(z)$. Repeating the arguments in the proof of Lemma 3 we obtain

$$\lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathbf{n}(\mathcal{A}_0(z), a) = \lim_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \left(\mathbf{n}(T^+(z), a) + \mathbf{n}(T^-(z), a) \right) = \frac{8}{3\pi a}. \quad (106)$$

\square

The proof of the following lemma uses the idea in [2].

Lemma 4. *Suppose that the integral operator functions $C_{1,2}(z) : \mathbb{R}_+/\{0\} \rightarrow \mathcal{C}(L^2(\mathbb{R}^2))$ have the integral kernels*

$$C_1(p, p') = \frac{z^2 g_z(|p|) g_z(|p'|)}{p^2 + p'^2 + p \cdot p' + z^2}, \quad (107)$$

$$C_2(p, p') = \frac{|p| |p'| g_z(|p|) g_z(|p'|)}{p^2 + p'^2 + p \cdot p' + z^2}. \quad (108)$$

Then $C_{1,2}(z) = \mathcal{O}_C(z)$.

Proof. By a direct check one finds that $\sup_{z>0} \|C_1(z)\|_{HS} < \infty$, hence $C_1(z) = \mathcal{O}_C(z)$ by Proposition 1. Consider the integral operator \tilde{C} on $L^2(\mathbb{R}^2)$ with the kernel

$$\tilde{C}(p, p') = \frac{\chi_{[0,1]}(|p|) \chi_{[0,1]}(|p'|)}{p^2 + p'^2}. \quad (109)$$

Let us show that this operator is bounded. Using the expansion like in (62)-(63) we reduce the problem to proving that the integral operator D on $L^2((0, 1); x dx)$ with the kernel

$$D(x, x') = \frac{1}{x^2 + x'^2} \quad (110)$$

is bounded. Consider the operator $W : L^2((0, 1); x dx) \rightarrow L^2(0, \infty)$, which acts on $f \in L^2((0, 1); x dx)$ according to the rule $[Wf](t) = e^{-t} f(e^{-t})$. The operator W has a bounded inverse and $\|Wf\| = \|f\|$, which means that W is unitary. The operator $WDW^{-1} : L^2(0, \infty) \rightarrow L^2(0, \infty)$ acts on $f(t)$ in the following way

$$[WDW^{-1}f](t) = \frac{1}{2} \int_0^\infty \frac{f(t') dt'}{\cosh(t - t')}. \quad (111)$$

Applying the Young inequality [15] we get

$$\|D\| = \|WDW^{-1}\| \leq \frac{1}{2} \int_0^\infty \frac{dx}{\cosh x} = \frac{\pi}{4}. \quad (112)$$

Now let us write $C_2(z)$ in the form

$$\begin{aligned} C_2(z) &= \chi_{(r, \infty)}(|p|) C_2(z) \chi_{(r, \infty)}(|p|) + \left\{ \chi_{[0, r]}(|p|) C_2(z) \chi_{(r, \infty)}(|p|) \right. \\ &\quad \left. + \chi_{(r, \infty)}(|p|) C_2(z) \chi_{[0, r]}(|p|) + \chi_{[0, r]}(|p|) C_2(z) \chi_{[0, r]}(|p|) \right\}. \end{aligned} \quad (113)$$

On one hand, $\chi_{(r, \infty)}(|p|) C_2(z) \chi_{(r, \infty)}(|p|) = \mathcal{O}_C(z)$ by Proposition 1. On the other hand, the norm of the terms in curly brackets can be made as small as pleased by choosing r small enough (this easily follows from (31) and the fact that \tilde{C} is bounded). Hence, $C_2(z) = \mathcal{O}_C(z)$. \square

IV. SPECTRAL BOUNDS FOR THE REMAINDER

Suppose that $A_{1,2} \in \mathcal{C}(\mathcal{H})$ and $a_{1,2} > 0$. Then the distribution function of singular values satisfies the inequality

$$\mathfrak{n}_\mu(A_1 + A_2, a_1 + a_2) \leq \mathfrak{n}_\mu(A_1, a_1) + \mathfrak{n}_\mu(A_2, a_2). \quad (114)$$

The proof of (114) can be found in [14] (see page 245). Using inequalities (1.4a), (1.4b) in [13] one can easily show that

$$\begin{aligned} \mathfrak{n}_\mu(AB, a) &\leq \mathfrak{n}_\mu(A, a\|B\|^{-1}), \\ \mathfrak{n}_\mu(BA, a) &\leq \mathfrak{n}_\mu(A, a\|B\|^{-1}) \end{aligned} \quad (115)$$

for any bounded B and $A \in \mathcal{C}(\mathcal{H})$. Following [13] we shall denote by J_p normed trace ideals of compact operators, recall that the norm of $A \in J_p$ is $\|A\|_p = (\sum_n \mu_n^p(A))^{1/p}$. The trace ideal J_2 is the family of Hilbert-Schmidt operators and $\|\cdot\|_2 \equiv \|\cdot\|_{HS}$. For $A \in J_p$, where $p \in [1, \infty)$

$$\mathfrak{n}_\mu(A, a) = \mathfrak{n}_\mu(A^*, a) \leq a^{-p} \|A\|_p^p. \quad (116)$$

Indeed,

$$\mathfrak{n}_\mu(A, a) = \mathfrak{n}_\mu(a^{-1}A, 1) \leq \sum_n \mu_n^p(a^{-1}A) = a^{-p} \|A\|_p^p. \quad (117)$$

Let us introduce the projection operator on \mathcal{H}_A

$$\mathbb{P}_\pm(z) = \text{diag} \left\{ \mathcal{F}_1^{-1} \mathbb{P}_\pm^{(1)}(z) \mathcal{F}_1, \mathcal{F}_2^{-1} \mathbb{P}_\pm^{(2)}(z) \mathcal{F}_2, \mathcal{F}_3^{-1} \mathbb{P}_\pm^{(3)}(z) \mathcal{F}_3 \right\}, \quad (118)$$

where $\mathbb{P}_\pm^{(k)}(z)$ act on $f(p_k, q_k)$ as follows

$$[\mathbb{P}_\pm^{(k)}(z)f](p_k, q_k) = \hat{\varphi}_\pm(\sqrt{z^2 + q_k^2}; p_k) \int \hat{\varphi}_\pm^*(\sqrt{z^2 + q_k^2}; p'_k) f(p'_k, q_k) dp'_k, \quad (119)$$

and $\hat{\varphi}_\pm(z; p)$ is the Fourier transform of $\varphi_\pm(z)$ in (12). Let us denote $\mathbb{P}(z) = \mathbb{P}_+(z) + \mathbb{P}_-(z)$ and $\mathbb{Q}(z) = 1 - \mathbb{P}(z)$, and similarly $P = P_+ + P_-$, where P_\pm were defined in (27). Using the cutoff operator in (19) we can write the decomposition

$$\mathcal{A}(z) = \tilde{\mathcal{A}}(z) + \mathcal{R}_1(z) + \mathcal{R}_1^*(z) + \mathcal{R}_2(z) + \mathcal{R}_3(z) + \mathcal{R}_3^*(z) + \mathcal{R}_4(z), \quad (120)$$

where

$$\begin{aligned}
\tilde{\mathcal{A}}(z) &= \mathfrak{X}_{[0, r_\varepsilon]} \mathbb{P}(z) \mathcal{A}(z) \mathbb{P}(z) \mathfrak{X}_{[0, r_\varepsilon]} \\
\mathcal{R}_1(z) &= \mathfrak{X}_{[0, r_\varepsilon]} \mathbb{Q}(z) \mathcal{A}(z) \mathbb{P}(z) \mathfrak{X}_{[0, r_\varepsilon]} \\
\mathcal{R}_2(z) &= \mathfrak{X}_{[0, r_\varepsilon]} \mathbb{Q}(z) \mathcal{A}(z) \mathbb{Q}(z) \mathfrak{X}_{[0, r_\varepsilon]} \\
\mathcal{R}_3(z) &= \mathfrak{X}_{[0, r_\varepsilon]} \mathcal{A}(z) \mathfrak{X}_{(r_\varepsilon, \infty)} \\
\mathcal{R}_4(z) &= \mathfrak{X}_{(r_\varepsilon, \infty)} \mathcal{A}(z) \mathfrak{X}_{(r_\varepsilon, \infty)}
\end{aligned}$$

The decomposition (32) holds true if we set

$$\mathcal{R}(z) = \sum_{k=0}^2 \mathcal{R}_{2k}(z) + \sum_{k=1}^2 [\mathcal{R}_{2k-1}(z) + \mathcal{R}_{2k-1}^*(z)], \quad (121)$$

where by definition

$$\mathcal{R}_0(z) = \tilde{\mathcal{A}}(z) - \mathcal{A}_0(z). \quad (122)$$

The proof of Theorem 3 is based on the following two lemmas

Lemma 5. *For all $\varepsilon > 0$ one can always fix $r_\varepsilon \in (0, 1/4)$ so that*

$$\overline{\lim}_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathfrak{n}_\mu(\mathcal{R}_i(z), \varepsilon) < \varepsilon \quad (i = 0, 1, 2). \quad (123)$$

Lemma 6. *For any fixed $r_\varepsilon \in (0, 1/4)$ and $\varepsilon > 0$*

$$\overline{\lim}_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathfrak{n}_\mu(\mathcal{R}_i(z), \varepsilon) = 0 \quad (i = 3, 4). \quad (124)$$

Proof of Theorem 3. By (121) and (114)

$$\mathfrak{n}_\mu(\mathcal{R}(z), \varepsilon) \leq \sum_{k=0}^2 \mathfrak{n}_\mu(\mathcal{R}_{2k}(z), \varepsilon/7) + \sum_{k=1}^2 2\mathfrak{n}_\mu(\mathcal{R}_{2k-1}(z), \varepsilon/7). \quad (125)$$

Let us fix r_ε as in Lemma 5. Then by Lemmas 5, 6

$$\overline{\lim}_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathfrak{n}_\mu(\mathcal{R}(z), \varepsilon) < 4\varepsilon/7. \quad (126)$$

□

We shall need the following estimates of the Hilbert-Schmidt operator norms

Lemma 7. *For $z \in (0, r_\varepsilon]$ and $R > 1$ there is $c > 0$ such that*

$$\|\mathfrak{X}_{[0, r_\varepsilon]} M'(z) G(z)\|_{HS}^2 \leq c r_\varepsilon^2 |\ln |\ln z^2||, \quad (127)$$

$$\|\mathfrak{X}_{[0, r_\varepsilon]} \mathbb{P}(z) \mathcal{A}(z) \mathfrak{X}_{(R, \infty)}\|_{HS}^2 \leq c R^{-2} |\ln |\ln z^2||. \quad (128)$$

Before proving Lemma 7 let us prove the following trivial bound

Lemma 8. *Suppose that $v_0 : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is Borel and $|v_0(x)| \leq \alpha_1 e^{-\alpha_2 |x|}$, where $\alpha_{1,2}$ are constants. Then its Fourier transform $\hat{v}_0(p)$ for any $a \in \mathbb{R}^2$ satisfies the inequality*

$$|\hat{v}_0(p+a) - \hat{v}_0(p-a)| \leq [\min(1, |a|)] |\hat{f}_a(p)|, \quad (129)$$

where $\hat{f}_a(p)$ is the Fourier-transform of $f_a \in L^2(\mathbb{R}^2)$ and $\sup_a \|f_a\| < \infty$.

Proof. By definition of the Fourier transform

$$\hat{v}_0(p+a) - \hat{v}_0(p-a) = \frac{1}{\pi} \int e^{-i(p \cdot x)} [e^{-i(a \cdot x)} - e^{i(a \cdot x)}] v_0(x) d^2x = [\min(1, |a|)] \hat{f}_a(p), \quad (130)$$

where

$$f_a(x) := -2[\min(1, |a|)]^{-1} \sin(a \cdot x) e^{-\frac{\alpha_2}{2}|x|} \left[e^{\frac{\alpha_2}{2}|x|} v_0(x) \right]. \quad (131)$$

Note that

$$\sup_{a \neq 0} \left\| [\min(1, |a|)]^{-1} \sin(a \cdot x) e^{-\frac{\alpha_2}{2}|x|} \right\|_{\infty} \leq \left\| (1 + |x|) e^{-\frac{\alpha_2}{2}|x|} \right\|_{\infty} < \infty. \quad (132)$$

The Lemma is proved because the norm of the function in square brackets in (131) is finite. \square

Proof of Lemma 7. Let us start with (127). Without loss of generality and in view of anti-symmetry relations (Table I) it is enough to prove that

$$\left\| (1 - p_{23}) \mathcal{F}_1^{-1} \chi_{[0, r_\varepsilon]}(|q_1|) \mathcal{F}_1 M_{12}(z) (1 - p_{13}) \mathcal{F}_2^{-1} g_z(|q_2|) \mathcal{F}_2 \right\|_{HS}^2 \leq c r_\varepsilon^2 |\ln |\ln z^2||, \quad (133)$$

for some $c > 0$, where the operator in (133) is considered on $L^2(\mathbb{R}^4)$. After applying the appropriate Fourier transform the operator in (133) has the following integral kernel (c.f. eq. (21))

$$\begin{aligned} K(p, q; p', q') &= \frac{1}{\pi^2} \chi_{[0, r_\varepsilon]}(|q|) \frac{g_z(|q'|)}{(2q' + q)^2 + 3q^2 + 3z^2} \\ &\times \left[\widehat{|v|^{\frac{1}{2}}} \left(p + \frac{2}{\sqrt{3}} q' + \frac{1}{\sqrt{3}} q \right) - \widehat{|v|^{\frac{1}{2}}} \left(p - \frac{2}{\sqrt{3}} q' - \frac{1}{\sqrt{3}} q \right) \right] \\ &\times \left[\widehat{|v|^{\frac{1}{2}}} \left(p' - \frac{1}{\sqrt{3}} q' - \frac{2}{\sqrt{3}} q \right) - \widehat{|v|^{\frac{1}{2}}} \left(p' + \frac{1}{\sqrt{3}} q' + \frac{2}{\sqrt{3}} q \right) \right]. \end{aligned} \quad (134)$$

Thus by Lemma 8

$$\begin{aligned} & \int |K(p, q; p', q')|^2 d^2 p d^2 q d^2 p' d^2 q' \\ & \leq c' \int \chi_{[0, r_\varepsilon]}(|q|) g_z^2(|q'|) \frac{|2q' + q|^2 |q' + 2q|^2}{[(2q' + q)^2 + 3q^2 + 3z^2]^2} d^2 q d^2 q' \leq c' (\alpha')^2 r_\varepsilon^2 \int g_z^2(|p|) d^2 p, \end{aligned} \quad (135)$$

where c', α' are constants and

$$\alpha' = \sup_{q', q \in \mathbb{R}^2} \frac{|2q' + q| |q' + 2q|}{(2q' + q)^2 + 3q^2 + 3z^2} < \infty. \quad (136)$$

On account of (31) for $z \leq r_\varepsilon$

$$\int g_z^2(|p|) d^2 p \leq (2\pi) \delta \int_0^{r_\varepsilon} \frac{t dt}{(t^2 + z^2) |\ln(t^2 + z^2)|} = \pi \delta [|\ln |\ln z^2|| - |\ln |\ln(z^2 + r_\varepsilon^2)||] \quad (137)$$

Substituting (137) into (135) we prove (127). Now let us consider (128).

$$\begin{aligned} \|\mathfrak{X}_{[0, r_\varepsilon]} \mathbb{P}(z) \mathcal{A}(z) \mathfrak{X}_{(R, \infty)}\|_{HS}^2 &= \left\| \mathbb{P}(z) G(z) M'(z) \mathfrak{X}_{(R, \infty)} (1 - M_d(z))^{-1/2} \right\|_{HS}^2 \\ &\leq c' \|G(z) M'(z) \mathfrak{X}_{(R, \infty)}\|_{HS}^2, \end{aligned} \quad (138)$$

where

$$c' = \sup_{z \in (0, r_\varepsilon]} \left\| (1 - M_d(z))^{-1/2} \mathfrak{X}_{(R, \infty)} \right\|^2 < \infty. \quad (139)$$

Hence, (128) would follow if we could prove that

$$\|\mathfrak{X}_{(R, \infty)} M'(z) G(z)\|_{HS}^2 \leq c R^{-2} |\ln |\ln z^2|| \quad (140)$$

for some $c > 0$ (in the last equation we have used that $\|A\|_{HS} = \|A^*\|_{HS}$ for $A \in \mathcal{C}(\mathcal{H})$).

Again without loss of generality the problem reduces to proving that

$$\|(1 - p_{23}) \mathcal{F}_1^{-1} \chi_{(R, \infty)}(|q_1|) \mathcal{F}_1 M_{12}(z) (1 - p_{13}) \mathcal{F}_2^{-1} g_z(|q_2|) \mathcal{F}_2\|_{HS}^2 \leq c R^{-2} |\ln |\ln z^2||, \quad (141)$$

for some $c > 0$. Let us denote by $\tilde{K}(p, q; p', q')$ the integral kernel of the operator on the lhs of (141). By Lemma 8 we obtain

$$\begin{aligned} & \int |\tilde{K}(p, q; p', q')| d^2 p d^2 q d^2 p' d^2 q' \\ & \leq \tilde{c} \int_{|q| > R} \frac{d^2 q}{[q^2 + z^2]^2} \int g_z^2(|q'|) d^2 q' \leq c R^{-2} |\ln |\ln z^2||, \end{aligned} \quad (142)$$

where $c, \tilde{c} > 0$ are constants. □

Proof of Lemma 5. Let us first consider $\mathcal{R}_0(z)$, which according to (122) can be rewritten as

$$\mathcal{R}_0(z) = \mathbb{P}(z)G(z)M'(z)G(z)\mathbb{P}(z) - PG(z)M'(z)G(z)P. \quad (143)$$

By (114) without losing generality it suffices to prove that for any $\varepsilon > 0$ one can choose r_ε so that

$$\overline{\lim}_{z \rightarrow 0} |\ln |\ln z^2||^{-1} \mathfrak{n}_\mu \left([\mathbb{P}(z) - P]G(z)M'(z)G(z)\mathbb{P}(z), \varepsilon \right) < \varepsilon/3. \quad (144)$$

Using the upper bound (116) we get

$$\begin{aligned} & \mathfrak{n}_\mu \left([\mathbb{P}(z) - P]G(z)M'(z)G(z)\mathbb{P}(z), \varepsilon \right) \\ & \leq \varepsilon^{-2} \left\| [\mathbb{P}(z) - P]G(z)M'(z)G(z)\mathbb{P}(z) \right\|_{HS}^2 \\ & \leq \varepsilon^{-2} \left\| [\mathbb{P}(z) - P]G(z) \right\|^2 \left\| \mathfrak{X}_{[0, r_\varepsilon]} M'(z)G(z) \right\|_{HS}^2 \end{aligned} \quad (145)$$

Note that from (13) it follows that $\|\varphi_\pm(z) - \eta_\pm\| \leq cz^2 |\ln z|$ for some $c > 0$. Together with (29)-(31) this gives

$$\overline{\lim}_{z \rightarrow 0} \left\| [\mathbb{P}(z) - P]G(z) \right\| \leq c'r_\varepsilon, \quad (146)$$

where $c' > 0$ is a constant. Now (123) for $i = 0$ follows from (145), (146) and Lemma 7 if we choose r_ε small enough. Now let us prove (123) for $i = 1$. By (116) and Lemma 7

$$\begin{aligned} \mathfrak{n}_\mu(\mathcal{R}_1(z), \varepsilon) & \leq \varepsilon^{-2} \left\| \mathbb{Q}(z)(1 - M_d(z))^{-\frac{1}{2}} \right\|^2 \left\| \mathfrak{X}_{[0, r_\varepsilon]} M'(z)G(z) \right\|_{HS}^2 \\ & \leq c_0 \varepsilon^{-2} r_\varepsilon^2 \left\| \mathbb{Q}(z)(1 - M_d(z))^{-\frac{1}{2}} \right\|^2 |\ln |\ln z^2||. \end{aligned} \quad (147)$$

From (16) it follows that

$$\overline{\lim}_{z \rightarrow 0} \left\| \mathbb{Q}(z)(1 - M_d(z))^{-\frac{1}{2}} \right\| < \infty \quad (148)$$

and thus (123) holds true for $i = 1$ if r_ε is chosen small enough. The proof of (123) for $i = 2$ is done analogously. \square

The proof of the next lemma is based on the following fact, which is proved on page 40 in [13]. Consider the operator $f(x)g(-i\nabla)$ acting on $L^2(\mathbb{R}^4)$, where $f, g \in L_3^2$ (for notations see [13]). Then

$$\|f(x)g(-i\nabla)\|_1 \leq C \left\{ \int (1 + |x|^2)^3 |f(x)|^2 d^4x \right\}^{\frac{1}{2}} \left\{ \int (1 + |x|^2)^3 |g(x)|^2 d^4x \right\}^{\frac{1}{2}}, \quad (149)$$

where the constant C does not depend on f, g .

Proof of Lemma 6. Let us first consider (124) for the case when $i = 4$. On one hand, by (115)

$$\mathfrak{n}_\mu(\mathcal{R}_4(z), \varepsilon) \leq \mathfrak{n}_\mu(\mathfrak{X}_{(r_\varepsilon, \infty)} M'(z), c^{-1} \varepsilon), \quad (150)$$

where

$$c = \sup_{z \in (0, r_\varepsilon]} \left\| (1 - M_d(z))^{-1/2} \mathfrak{X}_{(r_\varepsilon, \infty)} \right\|^2 < \infty. \quad (151)$$

On the other hand there is a constant $c' > 0$ such that

$$\left\| \mathfrak{X}_{(r_\varepsilon, \infty)} (M'(z) - M'(z')) \right\| \leq c' |z^2 - z'^2|. \quad (152)$$

Eq. (152) follows from (17) after the applying the resolvent identity. From (152) it follows that the operators $\mathfrak{X}_{(r_\varepsilon, \infty)} M'(z)$ form a Cauchy sequence for $z \rightarrow 0$ and converge in norm to a compact operator. Hence, the lhs of (150) is bounded by a constant for $z \in (0, r_\varepsilon]$ and (124) for $i = 4$ is proved. Now let us consider (124) for $i = 3$. By (114)

$$\mathfrak{n}_\mu(\mathcal{R}_3(z), \varepsilon) \leq \mathfrak{n}_\mu(\mathcal{R}_3^{(1)}(z), \varepsilon/3) + \mathfrak{n}_\mu(\mathcal{R}_3^{(2)}(z), \varepsilon/3) + \mathfrak{n}_\mu(\mathcal{R}_3^{(3)}(z), \varepsilon/3), \quad (153)$$

where

$$\begin{aligned} \mathcal{R}_3^{(1)}(z) &= \mathfrak{X}_{[0, r_\varepsilon]} \mathbb{P}(z) \mathcal{A}(z) \mathfrak{X}_{(r_\varepsilon, R]} \\ \mathcal{R}_3^{(2)}(z) &= \mathfrak{X}_{[0, r_\varepsilon]} \mathbb{Q}(z) \mathcal{A}(z) \mathfrak{X}_{(r_\varepsilon, \infty)} \\ \mathcal{R}_3^{(3)}(z) &= \mathfrak{X}_{[0, r_\varepsilon]} \mathbb{P}(z) \mathcal{A}(z) \mathfrak{X}_{(R, \infty)}, \end{aligned}$$

and $R \in (r_\varepsilon, \infty)$ is a parameter. Using the continuity arguments in the beginning of the proof one easily shows that

$$\overline{\lim}_{z \rightarrow 0} |\ln |\ln z^2|^{-1} \mathfrak{n}_\mu(\mathcal{R}_3^{(2)}(z), \varepsilon/3) = 0 \quad (154)$$

for all values of R . From Lemma 7 it follows that

$$\overline{\lim}_{z \rightarrow 0} |\ln |\ln z^2|^{-1} \mathfrak{n}_\mu(\mathcal{R}_3^{(3)}(z), \varepsilon/3) = o(1/R). \quad (155)$$

Thus instead of (124) for $i = 3$ it suffices to prove that

$$\overline{\lim}_{z \rightarrow 0} |\ln |\ln z^2|^{-1} \mathfrak{n}_\mu(\mathcal{R}_3^{(1)}(z), \varepsilon/3) = 0 \quad (156)$$

for all fixed $r_\varepsilon, \varepsilon, R > 0$. Like in the proof of Lemma 7 it suffices to show that

$$\overline{\lim}_{z \rightarrow 0} |\ln |\ln z^2|^{-1} \mathfrak{n}_\mu(\mathcal{K}(z), \varepsilon_0) = 0 \quad (157)$$

for all fixed $r_\varepsilon, \varepsilon_0, R > 0$, where

$$\mathcal{K}(z) = v_1 \mathcal{F}_1^{-1} g_z(|q_1|) \mathcal{F}_1 \mathcal{F}_2^{-1} (p_2^2 + q_2^2 + z^2)^{-1} \chi_{(r_\varepsilon, R]}(|q_2|) \mathcal{F}_2 v_2 \quad (158)$$

acts on the space $L^2(\mathbb{R}^4)$. Let us split $\mathcal{K}(z)$ into two parts $\mathcal{K}(z) = \mathcal{K}_1(z) + \mathcal{K}_2(z)$, where $\mathcal{K}_1(z) = \mathcal{K}(z) \chi_{[0, R']}(|x_1|)$ and $\mathcal{K}_2(z) = \mathcal{K}(z) \chi_{(R', \infty)}(|x_1|)$ and $R' > 0$ is a parameter. By (114) and (116)

$$\mathbf{n}_\mu(\mathcal{K}(z), \varepsilon_0) \leq 2\varepsilon_0^{-1} \|\mathcal{K}_1(z)\|_1 + 4\varepsilon_0^{-2} \|\mathcal{K}_2(z)\|_{HS}^2. \quad (159)$$

Using formula (149) we obtain the bound

$$\begin{aligned} \|\mathcal{K}_1(z)\|_1^2 &\leq c \|v\|_\infty^2 \int d^2x \int d^2y (1 + |x|^2 + |y|^2)^3 \chi_{[0, R']}(|x|) v^2 \left(\left| \frac{1}{2}x - \frac{\sqrt{3}}{2}y \right| \right) \\ &\times \int_{|p| \leq r_\varepsilon} \frac{d^2p}{(p^2 + z^2) |\ln(p^2 + z^2)|} \int_{r_\varepsilon \leq \left| \frac{\sqrt{3}}{2}p + \frac{1}{2}q \right| \leq R} d^2q \frac{(1 + p^2 + q^2)^3}{(p^2 + q^2 + z^2)^2} \leq c(R') |\ln |\ln z^2||, \end{aligned} \quad (160)$$

where $c(R') \in (0, \infty)$ is a constant, which depends on R' . (The first integral in (160) converges because v can be bounded by the exponent).

Let us write the Fourier transform \mathcal{F}_1 as a product $\mathcal{F}_1 = \mathcal{F}_x \mathcal{F}_y$, where $\mathcal{F}_{x,y}$ are partial Fourier transforms in variables x_1 and y_1 respectively. The operator $\mathcal{F}_y \mathcal{K}_2(z) \mathcal{F}_y^{-1}$ can be written as a product $\mathcal{F}_y \mathcal{K}_2(z) \mathcal{F}_y^{-1} = \mathcal{K}_2^{(1)}(z) \mathcal{K}_2^{(2)}(z)$, where

$$\begin{aligned} \mathcal{K}_2^{(1)}(z) &= \chi_{[0, r_\varepsilon]}(|q_1|) v(|x_1|) \mathcal{F}_x^{-1} (p_1^2 + q_1^2 + z^2)^{-1} \\ &\times \chi_{[r_\varepsilon, R]} \left(\left| \frac{\sqrt{3}}{2}p_1 + \frac{1}{2}q_1 \right| \right) \mathcal{F}_x \chi_{(R', \infty)}(|x_1|) \end{aligned} \quad (161)$$

$$\mathcal{K}_2^{(2)}(z) = g_z(|q_1|) \mathcal{F}_y v \left(-\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}y_1 \right) \mathcal{F}_y^{-1} \quad (162)$$

The integral operator $\mathcal{K}_2^{(1)}(z)$ has the kernel $\mathcal{K}_2^{(1)}(x_1, x'_1; q_1, z)$ and acts on $f(x_1, q_1) \in L^2(\mathbb{R}^4)$ as follows

$$[\mathcal{K}_2^{(1)}(z)f](x_1, q_1) = \int \mathcal{K}_2^{(1)}(x_1, x'_1; q_1, z) f(x'_1, q_1) d^2x'_1. \quad (163)$$

Similarly, $\mathcal{K}_2^{(2)}(z)$ has the kernel

$$\mathcal{K}_2^{(2)}(q_1, q'_1; x_1, z) = \frac{2}{3\pi} g_z(|q_1|) \exp \left(\frac{i}{\sqrt{3}} (q_1 - q'_1) \cdot x_1 \right) \hat{v}((2/\sqrt{3})(q_1 - q'_1)), \quad (164)$$

where \hat{v} is the Fourier transform of v . Now using (164) we can estimate the Hilbert-Schmidt norm

$$\|\mathcal{K}_2(z)\|_{HS}^2 = \|\mathcal{K}_2^{(1)}(z) \mathcal{K}_2^{(2)}(z)\|_{HS}^2 \leq \beta d(R') |\ln |\ln z^2||, \quad (165)$$

where $\beta > 0$ is a fixed constant and

$$d(R') = \sup_{\substack{|q_1| \leq r_\varepsilon \\ z \in [0, r_\varepsilon]}} \int |\mathcal{K}_2^{(1)}(x_1, x'_1; q_1, z)|^2 d^2 x_1 d^2 x'_1. \quad (166)$$

Let us show that $d(R') \rightarrow 0$ for $R' \rightarrow \infty$. Consider the compact integral operator $\mathcal{G}(q_1, z)$, which depends on the parameters q_1, z , acts on $L^2(\mathbb{R}^2)$ and has the structure $\mathcal{G}(q_1, z) = v(|x|)g(-i\nabla)$, where

$$g(s) = (s^2 + q_1^2 + z^2)^{-1} \chi_{[r_\varepsilon, R]} \left(\left| \frac{\sqrt{3}}{2}s + \frac{1}{2}q_1 \right| \right). \quad (167)$$

Then it is easy to see that

$$d(R') = \sup_{\substack{|q_1| \leq r_\varepsilon \\ z \in [0, r_\varepsilon]}} \left\| \mathcal{G}(q_1, z) \chi_{(R', \infty)}(|x|) \right\|_{HS}^2. \quad (168)$$

For fixed q_1, z the operator $\mathcal{G}(q_1, z)$ is Hilbert-Schmidt, this can be checked by using Eq. (4.7) in [13]. Thus for each fixed q_1, z the expression under supremum in (168) goes to zero for $R' \rightarrow \infty$. Using the same Hilbert-Schmidt norm estimate it is easy to show that for all $\epsilon > 0$ there is $\eta > 0$ such that $\|\mathcal{G}(q_1, z) - \mathcal{G}(q'_1, z')\|_{HS} < \epsilon/2$ if $|q_1 - q'_1| < \eta$ and $|z - z'| < \eta$. We can cover the set $\{q_1 | |q_1| \leq r_\varepsilon\} \cup \{z | z \in (0, r_\varepsilon]\}$ with the finite number of points $(q_1^{(i)}, z^{(i)})$ in such a way that for any $(q_1, z) \in \{q_1 | |q_1| \leq r_\varepsilon\} \cup \{z | z \in (0, r_\varepsilon]\}$ there would exist $(q_1^{(i_0)}, z^{(i_0)})$ such that $|q_1 - q_1^{(i_0)}| < \eta$ and $|z - z^{(i_0)}| < \eta$. Let us set R' so that $\max_i \|\mathcal{G}(q_1^{(i)}, z^{(i)}) \chi_{(R', \infty)}(|x|)\|_{HS} < \epsilon$. Then we would have $d(R') < \epsilon^2$. Since ϵ is arbitrary we conclude that $d(R') \rightarrow 0$ for $R' \rightarrow \infty$.

Summarizing, due to (159), (160) and (165) we have the upper bound

$$\mathbf{n}_\mu(\mathcal{K}(z), \varepsilon_0) \leq 2\varepsilon_0^{-1} [c(R')]^{\frac{1}{2}} |\ln |\ln z^2||^{\frac{1}{2}} + 4\varepsilon_0^{-2} \beta d(R') |\ln |\ln z^2||. \quad (169)$$

Thus it follows that the lhs in (157) is bounded by a fixed constant times $d(R')$. Letting $R' \rightarrow \infty$ we complete the proof of (124) for $i = 3$. \square

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